Notes on Spinning AdS_3 Black Hole Solution

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Abstract

By applying Newman's method, the AdS_3 rotating black hole solution is "derived" from the nonrotating black hole solution of Bañados, Teitelboim, and Zanelli (BTZ). The rotating BTZ solution derived in this fashion is given in "Boyer-Lindquist-type" coordinates whereas the form of the solution originally given by BTZ is given in a kind of an "unfamiliar" coordinates which are related to each other by a transformation of time coordinate alone. The relative physical meaning between these two time coordinates is carefully studied by evaluating angular momentum per unit mass, angular velocity, surface gravity and area of the event horizon in two alternative coordinates respectively. The result of this study leads us to the conclusion that the BTZ time coordinate must be the time coordinate of an observer who rotates around the axis of the spinning hole in opposite direction to that of the hole outside its static limit.

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I. Introduction

It had long been thought that black hole solutions cannot exit in 3-dim. since there is no local gravitational attraction and hence no mechanism to confine large densities of matter. It was, therefore, quite a surprise when Bañados, Teitelboim, and Zanelli (BTZ) [1] have recently constructed the Anti-de Sitter (AdS_3) spacetime solution to the Einstein equations in 3-dim. that can be interpreted as a black hole solution. They included the negative cosmological constant in the 3-dim. vacuum Einstein theory and then found both the rotating and nonrotating black hole solutions. In the mean time, a curious relationship between the nonrotating and the rotating spacetime solutions of Einstein theory in 4-dim. also has been long known. Newman et al. [2] discovered long ago that one can "derive" Kerr solution from the Schwarzschild solution in vacuum Einstein theory and Kerr-Newman solution from the Reissner-Nordström solution in Einstein-Maxwell theory via the "complex coordinate transformation" scheme acting on metrics written in terms of null tetrad of basis vectors. In the present work, we attempt the same derivation but in this time in 3-dim. spacetime. Namely, we see if the rotating version of BTZ black hole solution can indeed be "derived" from its nonrotating counterpart via Newman's method. And in doing so, our philosophy is that the 3-dim. situation can be thought of as, say, the $\theta = \pi/2$ - slice of the 4-dim. one (where θ denotes the polar angle). Interestingly enough, we do end up with the rotating version of BTZ black hole solution but in a different coordinate system from the one originally employed in BTZ's solution ansatz. And as we shall see shortly, it turns out that the rotating BTZ solution derived here in this work is given in "Boyer-Lindquist-type" [3] coordinates whereas the original form of the solution given by BTZ is given in a kind of an "unusual" coordinates which are related to the "familiar" Boyer-Lidquist-type coordinates by a transformation of time coordinate alone. We can easily understand the reason for this result as follows ; much as the Kerr solution "derived" from the Schwarzschild solution by Newman's complex coordinate transformation method naturally comes in Boyer-Lindquist coordinates [3] (of course via a coordinate transformation from Kerr coordinates [3]), the rotating BTZ black hole solution derived from its nonrotating counterpart in this manner

comes in Boyer-Lindquist-type coordinates as well. And then one can realize that by performing a transformation from the Boyer-Lindquist-type time coordinate t to a "new" time coordinate $\tilde{t}$ following the transformation law, $\tilde{t} = t - a\phi$ (where a is proportional to the angular momentum per unit mass and ϕ denotes the azimuthal angle), our "derived" rotating BTZ black hole solution can indeed be put in the form originally given by BTZ. As expected, the rotating BTZ black hole solution given in Boyer-Lindquist-type coordinates takes on the structure which resembles that of Kerr solution more closely than it is in BTZ's rather unusual time coordinate. However, one can readily realize that it is the BTZ's time coordinate $\tilde{t}$ that is the usual Killing time coordinate, not the Boyer-Lindquist- type one. And in order to investigate the relative physical meaning between these two time coodinates, quantities like angular momentum, angular velocity, surface gravity and area of the event horizon are evaluated in two alternative coordinates. The analysis of this observation on the behavior of the black hole solution with two different choices of coordinates will be presented later on in the discussion.

II. Derivation of the AdS$_3$ rotating black hole solution

As mentioned earlier, Newman et al. [2] discovered curious "derivations" of stationary, axisymmetric metric solutions from static, spherically-symmetric solutions in 4-dim. Einstein theory. To attempt the similar work in 3-dim. spacetime, we start with the nonrotating version of BTZ black hole solution in 3-dim. as a "seed" solution to construct its rotating counterpart. The nonrotating BTZ black hole solution written in Schwarzschild-type coordinates (t, r, ϕ) is given by

$$ds^2 = (-M + \frac{r^2}{l^2})dt^2 - (-M + \frac{r^2}{l^2})^{-1}dr^2 - r^2 d\phi^2 \tag{1}$$

where l is related to the negative cosmological constant by $l^{-2} = -\Lambda$ and M is an integration constant that can be identified with the ADM mass of the black hole. Now in order to "derive" a rotating black hole solution applying the complex coordinate transformation scheme of Newman et al., we begin by assuming that this 3-dim. black hole geometry is a $\theta = \pi/2$ - slice of a static, spherically-symmetric 4-dim. geometry given by

$$ds_4^2 = \lambda^2(r)dt^2 - \lambda^{-2}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

with $\lambda^2(r) = (-M + r^2/l^2)$. The essential reason for this "dimensional continuation" is to introduce the null tetrad system of vectors on which Newman's complex coordinate transformation method is technically based. We do not, however, ask nor demand that this 4-dim. geometry be an explicit solution of Einstein equation in 4-dim. as well. We just demand that only its $\theta = \pi/2$ - slice be a solution of Einstein equation in 3-dim. Remarkably, then, upon the series of operations ; dimensional continuation $\rightarrow$ Newman's derivation method $\rightarrow$ dimensional reduction by setting $\theta = \pi/2$, we end up with a legitimate rotating black hole solution to 3-dim. Einstein equation as we shall see shortly.

Consider now the transformation to the Eddington-Finkelstein-type retarded null coordinates (u, r, ϕ) defined by $u = t - r_*$, with $r_* = \int dr(g_{rr}/ - g_{tt})^{1/2}$. In terms of these null coordinates, the nonrotating BTZ black hole metric takes the form

$$ds_4^2 = \lambda^2(r)du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3}$$

Into this 4-dim. Riemannian space we next introduce a tetrad system of vectors l_μ, n_μ, m_μ and $\bar{m}_\mu$ (with $\bar{m}_\mu$ being the complex conjugate of m_μ) satisfying the following orthogonality property $l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1$ with all other scalar products vanishing. In terms of this null tetrad of basis vectors, the spacetime metric is written as $g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu$. Then now one can obtain the contravariant components of the metric and the null tetrad vectors from the covariant components of the metric given in eq.(3) as

$$\begin{aligned} g^{00} &= 0, \quad g^{11} = -\lambda^2(r), \quad g^{10} = 1, \\ g^{22} &= -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2\sin^2\theta} \end{aligned} \tag{4}$$

and

$$\begin{aligned} l^\mu &= \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \frac{1}{2}\lambda^2(r)\delta_1^\mu, \\ m^\mu &= \frac{1}{\sqrt{2}r}(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu), \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}r}(\delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu) \end{aligned} \tag{5}$$

respectively. Now the radial coordinate r is allowed to take complex values and the tetrad is rewritten in the form

$$
\begin{aligned}
l^\mu &= \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \frac{1}{2}(-M + \frac{r\bar{r}}{l^2})\delta_1^\mu, \\
m^\mu &= \frac{1}{\sqrt{2}\bar{r}}(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu), \\
\bar{m}^\mu &= \frac{1}{\sqrt{2}r}(\delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu)
\end{aligned} \tag{6}
$$

with $\bar{r}$ being the complex conjugate of r (note that part of the algorithm is to keep l^μ and n^μ real and m^μ and $\bar{m}^\mu$ the complex conjugate of each other). We now formally perform the "complex coordinate transformation"

$$
\begin{aligned}
r' &= r + ia\cos\theta, \quad \theta' = \theta, \\
u' &= u - ia\cos\theta, \quad \phi' = \phi
\end{aligned} \tag{7}
$$

on tetrad vectors l^μ, n^μ and m^μ ($\bar{m}'^\mu$ is, as stated, defined as the complex conjugate of m'^μ). If one now allows r' and u' to be real, we obtain the following tetrad

$$
\begin{aligned}
l'^\mu &= \delta_1^\mu, \\
n'^\mu &= \delta_0^\mu - \frac{1}{2}[-M + \frac{(r'^2 + a^2\cos^2\theta)}{l^2}]\delta_1^\mu, \\
m'^\mu &= \frac{1}{\sqrt{2}(r' + ia\cos\theta)}[ia\sin\theta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu]
\end{aligned} \tag{8}
$$

from which one can readily read off the contravariant components of the metric and then obtain the covariant components by inversion as (henceforth we drop the "prime")

$$
\begin{aligned}
ds_4^2 &= (-M + \frac{\Sigma}{l^2})du^2 + 2a\sin^2\theta[1 - (-M + \frac{\Sigma}{l^2})]dud\phi + 2dudr \\
&\quad - 2a\sin^2\theta drd\phi - \Sigma d\theta^2 - [r^2 + a^2 + a^2\sin^2\theta\{1 - (-M + \frac{\Sigma}{l^2})\}]\sin^2\theta d\phi^2
\end{aligned} \tag{9}
$$

where $\Sigma \equiv (r^2 + a^2\cos^2\theta)$. Now at this stage, considering that the geometry in 3-dim. can be thought of as the $\theta = \pi/2$-slice of the full, 4-dim. one, we simply set $\theta = \pi/2$ in the metric above to arrive at the rotating black hole metric in 3-dim. given by

$$
ds^2 = (-M + \frac{r^2}{l^2})(du - ad\phi)^2 + 2(du - ad\phi)(dr + ad\phi) - r^2 d\phi^2. \tag{10}
$$

Also note that the metric above we "derived" via Newman's complex coordinate transformation method is given in terms of Kerr-type coordinates [3] (u, r, ϕ) which can be thought of as the generalization of the retarded null coordinates. Thus one might want to further transform it into the one written in Boyer-Lindquist-type coordinates [3] $(t, r, \hat{\phi})$ that can be viewed as the generalization of the Schwarzschild coordinates. This can be achieved via the transformation

$$dt = du + \frac{(r^2 + a^2)}{\Delta} dr, \quad d\hat{\phi} = d\phi + \frac{a}{\Delta} dr \tag{11}$$

where $\Delta \equiv r^2(-M + r^2/l^2) + a^2$. Finally, the rotating AdS_3 black hole solution given in Boyer-Lindquist- type coordinates is given by (henceforth we drop "hat" on ϕ coordinate)

$$\begin{aligned} ds^2 = &(-M + \frac{r^2}{l^2})dt^2 + 2a[1 - (-M + \frac{r^2}{l^2})]dt d\phi \\ &- [r^2 + a^2 + a^2\{1 - (-M + \frac{r^2}{l^2})\}]d\phi^2 - \frac{r^2}{\Delta}dr^2. \end{aligned} \tag{12}$$

And it is straightforward to check that the "derived" rotating black hole solution given in eq.(12) does satisfy the AdS_3 Einstein equation, $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - l^{-2}g_{\mu\nu} = 0$.

It is a little puzzling, however, that the derived rotating black hole solution of ours in eq.(12) above does not appear exactly the same as the one originally constructed by BTZ. Nonetheless, both our solution above and the one originally obtained by BTZ (written in our notation convention in which $a = J/2$ with J appearing in original BTZ's work [1])

$$ds^2 = (-M + \frac{r^2}{l^2})d\tilde{t}^2 + 2a d\tilde{t} d\phi - r^2 d\phi^2 - \frac{r^2}{\Delta}dr^2 \tag{13}$$

correctly reduce, in the vanishing angular momentum limit $a \to 0$, to the static, spherically-symmetric black hole solution which was the starting point of our solution construction. Therefore, presumably the two rotating black hole solutions must be related to each other by a coordinate transformation. Indeed, one can check straightforwardly that the two are related by the transformation of the time coordinate alone

$$\tilde{t} = t - a\phi. \tag{14}$$

Therefore the two metrics given in eqs.(12) and (13) represent one and the same AdS_3 black hole solution modulo gauge transformation. Notice, however, that the rotating BTZ black hole solution given in the Boyer-Lindquist-type time coordinate in eq.(12) resembles the structure of Kerr solution in 4-dim. more closely than that given in the BTZ time coordinate in eq.(13). It seems now that we are left with the question ; between t and $\tilde{t}$, which one is the usual Killing time coordinate ? It appears that it is $\tilde{t}$, the BTZ time coordinate that is the usual Killing time coordinate, *not* t, the Boyer-Lindquist-type time coordinate. And this conclusion is based on the following observation. Note that both in Boyer-Lindquist-type coordinates $(t,\ r,\ \phi)$ and in BTZ coordinates $(\tilde{t},\ r,\ \phi)$, generally $\mid g_{t\phi} \mid$ $(\mid g_{\tilde{t}\phi} \mid)$ represents *angular momentum per unit mass* as observed by an accelerating observer at some fixed r. (We stress here that the quantity we are about to introduce is the angular momentum *per unit mass* at some point r. Generally, it should be distinguished from the total angular momentum of the spacetime measured in the asymptotic region, $\tilde{J} = (16\pi)^{-1} \int_S \epsilon_{\mu\nu\alpha\beta} \nabla^\alpha \psi^\beta$ in the notation convention of ref.[4] with S being a large sphere in the asymptotic region and $\psi^\mu = (\partial/\partial\phi)^\mu$ being the rotational Killing field. Thus in these definitions, angular momentum *per unit mass* may change under a coordinate transformation although the total angular momentum $\tilde{J}$ remains coordinate independent.) To see this quickly, one only needs to write the rotating black hole metric in ADM's (2+1)-split form

$$ds^2 = N^2(r)dt^2 - f^{-2}(r)dr^2 - R^2(r)[N^\phi(r)dt + d\phi]^2. \tag{15}$$

Then in Boyer-Lindquist-type time coordinate t, metric components correspond to

$$\begin{aligned} f^2(r) &= (-M + \frac{r^2}{l^2} + \frac{a^2}{r^2}), \\ R^2(r) &= [r^2 + a^2 + a^2\{1 - (-M + \frac{r^2}{l^2})\}], \\ N^2(r) &= (-M + \frac{r^2}{l^2}) + R^{-2}(r)a^2\{1 - (-M + \frac{r^2}{l^2})\}^2, \\ N^\phi(r) &= -R^{-2}(r)a\{1 - (-M + \frac{r^2}{l^2})\} \end{aligned} \tag{16}$$

whereas in BTZ time coordinate $\tilde{t}$, they correspond to

$$N^2(r) = f^2(r) = (-M + \frac{r^2}{l^2} + \frac{a^2}{r^2}),$$
$$N^\phi(r) = -\frac{a}{r^2}, \quad R^2(r) = r^2.$$

Now it is apparent in this ADM's space-plus-time split form of the metric given in eq.(15) that the shift function $N^\phi(r)$ corresponds to the angular velocity $\mid N^\phi(r) \mid = \Omega(r)$ and the quantity $R^2(r) \mid N^\phi(r) \mid = J$ can be identified with the angular momentum per unit mass as observed at some point r outside the rotating hole.

Thus from $-R^2(r)N^\phi(r) = g_{t\phi}$, the angular momentum per unit mass in Boyer-Lindquist-type time coordinate and in BTZ time coordinate are given respectively by

$$J = \mid g_{t\phi} \mid = a[1 - (-M + \frac{r^2}{l^2})],$$
$$J^{BTZ} = \mid g_{\tilde{t}\phi} \mid = a \tag{17}$$

and thus in particular, the angular momentum per unit mass at the event horizon is given respectively by

$$J_H = \mid g_{t\phi}(r_+) \mid = a\frac{(r_+^2 + a^2)}{r_+^2},$$
$$J_H^{BTZ} = \mid g_{\tilde{t}\phi}(r_+) \mid = a \tag{18}$$

indicating that in BTZ time coordinate the spinning hole appears to have smaller angular momentum per unit mass at the event horizon (and generally everywhere outside the hole) than it does in Boyer-Lindquist-type time coordinate. Now from eq.(17), it is manifest that the angular momentum per unit mass given in BTZ time coordinate $\tilde{t}$ is finite and constant all over the hypersurface whereas that given in Boyer-Lindquist-type time coordinate t grows indefinitely as $r \to \infty$. This can be attributed to the fact that the coordinate $\tilde{t}$ BTZ used is defined asymptotically using the asymptotic symmetries and hence approaches the AdS time at spatial infinity [1]. Therefore we can conclude that it is the BTZ time coordinate $\tilde{t}$ which is the usual Killing time. This is certainly in contrast to what happens in the familiar Kerr black hole geometry in 4-dim. where the usual Boyer-Lindquist time coordinate is the Killing time coordinate. And it seems that this discrepancy comes from the fact that the

3-dim. BTZ black hole is not asymptotically flat but asymptotically anti-de Sitter whereas the 4-dim. Kerr black hole is asymptotically flat. The final question one might want to ask and answer could then be ; what is the relative physical meaning of these two time coordinates t and $\tilde{t}$? To get a quick answer to this question, we go back and look at the coordinate transformation law given in eq.(14) relating the two time coordinates t and $\tilde{t}$. Namely, taking the dual of the transformation law $\delta\tilde{t} = \delta t - a\delta\phi$, we get

$$(\frac{\partial}{\partial\tilde{t}})^{\mu} = (\frac{\partial}{\partial t})^{\mu} - \frac{1}{a}(\frac{\partial}{\partial\phi})^{\mu}$$

or

$$\tilde{\xi}^{\mu} = \xi^{\mu} - \frac{1}{a}\psi^{\mu}$$

where $\xi^{\mu} = (\partial/\partial t)^{\mu}$ and $\psi^{\mu} = (\partial/\partial\phi)^{\mu}$ denote Killing fields corresponding to the time translational and the rotational isometries of the spinning black hole spacetime respectively and $\tilde{\xi}^{\mu} = (\partial/\partial\tilde{t})^{\mu}$ denotes the Killing field associated with the isometry of the hole's metric under the BTZ time translation. Now this expression for the BTZ time translational Killing field $\tilde{\xi}^{\mu}$ implies that in BTZ time $\tilde{t}$, the time translational generator is given by the linear combination of the Boyer-Lindquist-type time translational generator and the rotational generator. In plain English, this means that in BTZ time coordinate, the action of time translation consists of the action of Boyer-Lindquist-type time translation and the action of rotation in opposite direction to a, i.e., to the rotation direction of the hole. Thus the BTZ time coordinate $\tilde{t}$ can be interpreted as the coordinate, say, of a frame which rotates around the axis of the spinning BTZ black hole in opposite direction to that of the hole. Furthermore, by considering the angular velocity, the angular momentum per unit mass, the surface gravity and the area of the event horizon of the hole both in Boyer-Lindquist-type time and in BTZ time coordinate and then comparing them, one can explore the relative physical meaning between the two time coordinates in a more comprehensive manner. And to do so, we need to compare the causal structures and the black hole thermodynamics investigated in the two alternative time coordinates. (The reason for doing this is, as we shall see, aside from the value of studying causal structure and thermodynamics themselves,

they also will provide us with very clear and natural interpretation of the two alternative time coordinates t and $\tilde{t}$.) Since they have already been studied in the BTZ time coordinate $\tilde{t}$ by BTZ in their original works, here in this work we consider them in the Boyer-Lindquist-type time coordinate. As we shall see shortly, the causal structure and the global topology remain unaffected under the coordinate change $\tilde{t} = t - a\phi$ as they shoud. An important point, however, is that quantities like angular momentum per unit mass, angular velocity and the surface gravity at the event horizon are obtained *differently* in the two alternative time coordinates. And it is precisely this point that will enable us to realize the relative physical meaning of the two time coordinates as we shall see in the following section.

III. Causal structure and black hole thermodynamics

Now consider the causal structure [4] of the rotating BTZ black hole solution in the Boyer-Lindquist-type time coordinates. As we shall see shortly, it turns out that its causal structure is almost the same as that in the BTZ time coordinate and their global structures are exactly the same as is manifest in their Carter-Penrose diagrams (we shall not provide the diagrams here since they can be found in the literature). First we start with the curvature singularity. Inspection of the behavior of typical curvature invariant shows that it is everywhere regular including $r = 0$. In fact, this was expected since generally the (anti-)de Sitter spacetime is a maximally-symmetric space with a constant curvature. Next we consider the event horizons. As mentioned earlier, our rotating AdS_3 black hole solution is stationary and axisymmetric and thus possesses two Killing fields $\xi^{\mu} = (\partial/\partial t)^{\mu}$ and $\psi^{\mu} = (\partial/\partial\phi)^{\mu}$ correspondingly. And it is their linear combination $\chi^{\mu} = \xi^{\mu} + \Omega_H \psi^{\mu}$ which is normal to the Killing horizons of the black hole spacetime. In fact, normaly this is the defining equation of the angular velocity of the event horizon, Ω_H. Now since the Killing horizon is defined to be a surface on which the Killing field χ^{μ} becomes null, in order to find the event horizon we should look for zeros of $\chi^{\mu}\chi_{\mu} = 0$. A straightforward calculation shows that the Killing field χ^{μ} becomes null at points where $\Delta = r^2(-M + r^2/l^2) + a^2 = 0$. Thus we have regular inner and outer horizons at

$$r_{\pm} = l[\frac{M}{2}\{1 \pm \sqrt{1 - (\frac{2a}{Ml})^2}\}]^{1/2} \tag{19}$$

(i.e., $M = (r_+^2 + r_-^2)/l^2$ and $a = r_+ r_-/l$) with r_+being the black hole event horizon provided $|a| \leq Ml/2$. Note that these coordinate singularities are absent in Kerr-type null coordinates although manifest in Boyer-Lindquist-type coordinates. And the angular velocity of this event horizon is given by

$$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}}|_{r_+} = \frac{a}{(r_+^2 + a^2)}. \tag{20}$$

Also note that rotating BTZ black hole solution given in Boyer-Lindquist-type time coordinate develops inner and outer horizons exactly at the same locations as those in BTZ time coordinate. However, there is an important difference ; the angular velocity of the event horizon evaluated in Boyer-Lindquist-type time coordinate, $\Omega_H = a/(r_+^2 + a^2)$ is smaller than that in BTZ time coordinate

$$\Omega_H^{BTZ} = -\frac{g_{\tilde{t}\phi}}{g_{\phi\phi}}|_{r_+} = \frac{a}{r_+^2} \tag{21}$$

and thus the hole appears to *spin more slowly* in Boyer-Lindquist-type time coordinate. Finally, we consider the "static limit" which is the outer boundary of the ergoregion. Note that the time-translational Killing field with the norm $\xi^\mu \xi_\mu = g_{tt} = (-M + r^2/l^2)$ becomes spacelike, null and then timelike as r increases. Particularly note that the region in which ξ^μ stays spacelike extends outside the black hole's event horizon. This region is usually called "ergoregion" and its outer boundary on which ξ^μ becomes null is called "static limit" since inside of which no observer can possibly remain static. Thus in order to find the location of this static limit, we look for a zero of $\xi^\mu \xi_\mu = 0$. And it turns out that the static limit of the rotating BTZ black hole solution in Boyer-Lindquist-type coordinate occurs at $r_s = \sqrt{M}l > r_+$ again exactly at the same location as that in BTZ time coordinate.

Finally, we discuss the thermodynamics of the rotating black hole solution given in Boyer-Lindquist-type time coordinate. Since the practical study of black hole thermodynamics [4] begins and ends with the temperature and entropy of the hole, we shall attempt to compute

them. We first begin with the Hawking temperature T_H measured by an observer in the asymptotic region. The Hawking temperture T_H is related to the surface gravity κ of the black hole by the relation $T_H = \kappa/2\pi$ which is supposed to hold for any stationary black hole [6]. Thus our task is simply to calculate the surface gravity of the hole. In physical terms, the surface gravity κ is the force that must be exerted to hold a unit test mass at the horizon and it is given in a simple formula as [4]

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu \chi^\nu)(\nabla_\mu \chi_\nu) \tag{22}$$

where χ^μ is as given earlier and the evaluation on the event horizon $r = r_+$ is assumed. A direct computation then gives the Hawking temperature as

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{2\pi l^2}\frac{r_+(r_+^2 - r_-^2)}{(r_+^2 + a^2)} = \frac{1}{2\pi r_+}(\frac{r_+^2 - r_-^2}{r_-^2 + l^2}) \tag{23}$$

where we used $M = (r_+^2 + r_-^2)/l^2$ and $a = r_+ r_-/l$. Recall that the Hawking temperature (or surface gravity) of the rotating BTZ solution in BTZ time coordinate is given by $T_H^{BTZ} = (r_+^2 - r_-^2)/2\pi l^2 r_+$. Next, we turn to the computation of the black hole's entropy. Generally, the black hole entropy (i.e., the semiclassical entropy, not the fine-grained, quantum statistical one) can be evaluated in three ways : first, following Bekenstein-Hawking proposal [5, 6], one can argue *a priori* that the entropy of a black hole must be proportional to the surface area of its event horizon ($S = A/4$). Alternatively, knowing the Hawking temperature T_H and the chemical potential (i.e., Ω_H for the conserved angular momentum J_H), one may integrate the 1st law of black hole thermodynamics $T_H dS = dM - \Omega_H dJ_H +$ (variation of gravitational source terms), to obtain the entropy S. Lastly, according to Gibbons and Hawking [7], thermodynamic functions including the black hole entropy can be computed directly from the saddle point approximation to the gravitational partition function $S = \ln Z + \beta(H - \Omega_H J_H) \simeq -I[g^c] + (M - \Omega_H J_H)/T_H$ where $I[g^c]$ is the Euclidean action evaluated at the saddle point g^c, i.e., the classical metric solution and Z is the partition function. Although the second method turns out to be inappropriate for asymptotically non-flat cases like this AdS$_3$ black hole case, the first and the third methods, of course, agree to yield the same expression for the entropy as

$$S = \frac{1}{4}A = \frac{\pi(r_+^2 + a^2)}{2r_+} = \frac{\pi r_+}{2l^2}(r_-^2 + l^2) \tag{24}$$

where again we used $a = r_+ r_-/l$. For the sake of comparison, recall that the entropy of the rotating BTZ solution in BTZ time coordinate is given by $S^{BTZ} = \pi r_+/2$.

Now we summarize what we have learned concerning the causal structures and the black hole thermodynamics investigated in the two alternative time coordinates. It turned out that whichever time coordinate, the Boyer-Lindquist-type one t or the BTZ one $\tilde{t}$, one may take, one ends up with the same causal structure and the global topology except for the fact,

$$\begin{aligned} J_H &= (\frac{r_+^2 + a^2}{r_+^2})J_H^{BTZ} > J_H^{BTZ}, \\ \Omega_H &= (\frac{r_+^2}{r_+^2 + a^2})\Omega_H^{BTZ} < \Omega_H^{BTZ}, \\ T_H &= (\frac{r_+^2}{r_+^2 + a^2})T_H^{BTZ} < T_H^{BTZ}, \\ S &= (\frac{r_+^2 + a^2}{r_+^2})S^{BTZ} > S^{BTZ} \end{aligned} \tag{25}$$

which are obtained from eqs.(18), (20), (21), (23) and (24) and where quantities with the superscript "BTZ" denote the ones computed in BTZ time coordinate $\tilde{t}$, whereas quantities without superscript denote the ones computed in Boyer-Lindquist-type time coordinate t.

These results indicate that in BTZ time coordinate $\tilde{t}$, the rotating BTZ black hole solution has smaller angular momentum per unit mass yet greater angular velocity, greater surface gravity and smaller area of the event horizon than it does in the Boyer-Lindquist-type time coordinate t. Particularly here, "possessing smaller angular momentum per unit mass while greater angular velocity" may first look erroneous but if one really looks into the details one can realize that it is no nonsense since it arises from "same coordinate distance but different proper distances" to the horizon in the two time coordinates t and $\tilde{t}$.

Now, first from the greater angular velocity, we are led to the conclusion that the BTZ time coordinate $\tilde{t}$ must be the one, say, of an observer who *rotates around the axis of the spinning hole in opposite direction to the rotation direction of the hole outside its static*

limit. This interpretation, then, readily explains the smaller length of the horizon (horizon area becomes *length* in 3-dim.) as being the Lorentz length contraction. Next, smaller angular momentum per unit mass and greater surface gravity can be attributed to the fact that as the angular momentum per unit mass decreases when transforming from the Boyer-Lindquist-type to BTZ time coordinate, the surface gravity is expected to increase due to an effect of centrifugal force.

IV. Discussions

To conclude, here in this work we attempted to apply Newman's method of generating a spinning black hole solution from a static black hole solution in 4-dim. Einstein theory to the 3-dim. situation. More concretely, we employed the algorithm ; dimensional continuation $\rightarrow$ Newman's derivation method $\rightarrow$ dimensional reduction by setting $\theta = \pi/2$ to successfully obtain a legitimate rotating AdS_3 black hole solution. And as a consequence of this study, we discovered that there are two alternative time coordinates in describing the rotating AdS_3 black hole solution one can select from to investigate its various physical contents. Thus let us elaborate on the complementary roles played by the two alternative time coordinates. First, note that the two time coordinates, that of Boyer-Lindquist-type, t and that of BTZ, $\tilde{t}$ coincide for nonrotating case ($a = 0$) and become different only for rotating case ($a \neq 0$) as one can see in their relation, $\tilde{t} = t - a\phi$. Next, the Boyer-Lindquist-type time coordinate t is the usual time coordinate with which one can obtain the rotating black hole's characteristics such as the angular velocity of the event horizon or the surface gravity as measured by an outside observer who is "static" with respect to, say, a distant body just as it is the case with Boyer-Lindquist time coordinate in Kerr black hole spacetime in 4-dim. The BTZ time coordinate $\tilde{t}$, on the other hand, may look "unusual" in that it can be identified with the time coordinate of a non-static observer who rotates in opposite direction to that of the spinning hole. It is, however, this BTZ time coordinate, not the familiar Boyer-Lindquist-type time coordinate, which is the right Killing time coordinate in terms of which the total mass and particularly the angular momentum can be well-defined in the asymptotic region of this asymptotically anti-de Sitter spacetime. Also it has been well-studied in the literature [1]

that this BTZ time coordinate is particularly advantageous in exploring the global structure of the rotating BTZ black hole since it allows one to transform to Kruskal-type coordinates and eventually allows one to draw the Carter-Penrose conformal diagram much more easily than the case when one employs the usual Boyer-Lindquist-type time coordinate. Therefore when investigating various physical contents of the spinning BTZ black hole, the two time coordinates t and $\tilde{t}$ appear to play mutually complementary roles.

References

[1] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992) ; M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. **D48**, 1506 (1993).

[2] E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915 (1965) ; E. T. Newman, E. Couch, R. Chinnapared, A. Exton, A. Prakash and R. Torrence, J. Math. Phys. **6**, 918 (1965).

[3] R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 265 (1967) ; B. Carter, Phys. Rev. **174**, 1559 (1968).

[4] R. M. Wald, *General Relativity* (Univ. of Chicago Press, Chicago, 1984).

[5] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973) ; *ibid*, **D9**, 3292 (1974).

[6] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[7] G. W. Gibbons and S. W. Hawking, Phys. Rev. **D15**, 2752 (1977).